Minimal solution of the AdS/CFT crossing equation

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Abstract
We solve explicitly the crossing equation under sufficiently general assumptions
on the structure of the dressing phase. We obtain the BES/BHL dressing phase
as a minimal solution of the crossing equation and identify the possible CDD
factors.

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1. Introduction

There is evidence that the spectral problem which appears in the context of the AdS/CFT
correspondence can be solved exactly under the assumption of integrability [1–3]. In analogy
with the bootstrap approach in the relativistic theories [4], an important object to identify is a
factorized scattering matrix [5]. The knowledge of the scattering matrix leads to the solution
of the spectral problem of the theory in a large volume in terms of the asymptotic Bethe Ansatz
proposed in [6, 7]. From the symmetries of the theory it is possible to fix the scattering matrix
up to a scalar factor [7, 8], also known as a dressing factor.

The necessity of the nontrivial dressing factor was first observed in [9]. Later it was shown
that it is constrained by the crossing equations [10]. A proposal for the logarithm of the scalar
factor, known as the BES/BHL dressing phase, was given in [11–13]. This proposal passed
many nontrivial checks. In particular, it has a correct strong coupling asymptotics which was
found at tree level [9] and at one loop [14–16] using the method of the algebraic curve solution
[17]. The conjectured dressing phase was an important ingredient for the so-called BES
equation [13, 18]. The solution of the BES equation at weak coupling [13] is in agreement
with explicit four-loop perturbative calculations in $\mathcal{N} = 4$ SYM [19]. The strong coupling
solution of the BES equation [20, 21] is in agreement with a two-loop string prediction [22].
Now it is widely accepted that the BES/BHL dressing phase gives a correct scalar factor of
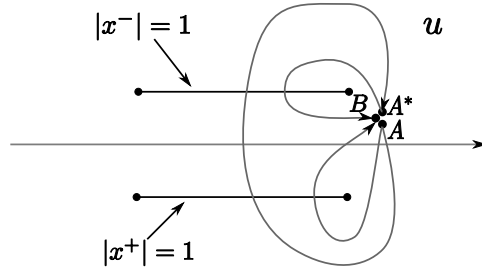the AdS/CFT scattering matrix.

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Figure 1. The crossing A^* and the mirror points B in the u-plane.

In integrable relativistic field theories the solution of the crossing equation is not unique. The correct scalar factor is usually a 'minimal' solution of the crossing equation, i.e. the solution with a minimal number of singularities in the physical strip [4]. All other solutions of the crossing equation can be obtained via multiplication by the CDD factors.

In this paper, we argue that this is the case also for the AdS/CFT integrable system. We explicitly solve the crossing equation and show that the 'minimal' solution coincides with the BES/BHL dressing factor. We also comment on the possible form of the CDD factors.

2. Crossing relation at the mirror point

In the following, we will use the Jukowsky map $x[u]$ defined by

$$x = \frac{u}{2g}\left(1 + \sqrt{1 - \frac{4g^2}{u^2}}\right), \qquad \frac{u}{g} = x + \frac{1}{x}. \tag{1}$$

We introduce the following shorthand notations. By x, y and z we denote respectively the images of u, v and w. We also denote $x^{\pm} \equiv x[u \pm i/2]$ and $y^{\pm} \equiv x[v \pm i/2]$. We take the branch of the Jukowsky map such that $|x| > 1$ if the opposite is not mentioned explicitly.

The crossing equation is formulated as follows [10, 24]:

$$\sigma[u, v]\sigma^{\mathrm{cross}}[u, v] = \frac{y^-}{y^+}\frac{x^- - y^+}{x^+ - y^+}\frac{1 - \frac{1}{x^- y^-}}{1 - \frac{1}{x^+ y^-}}, \tag{2}$$

where $\sigma[u, v]$ is a dressing factor and $\sigma^{\mathrm{cross}}[u, v]$ is its crossing transform.

The dressing factor is a multivalued function of u and v. In the following, we will fix v and consider σ as a function of u.

The crossing transformation acts as the analytical continuation of σ along the contour AA^* in the u-plane, depicted in figure 1, which encircles the branch points $2g \pm i/2$. As is seen from (2), the dressing factor σ has a nontrivial monodromy along the contour AA^*.

It is possible to write the crossing equation as a periodicity condition using the elliptic uniformization [10]. In terms of the elliptic variable s used in [23] and related to one in [10] by a Gauss–Landen transformation, the crossing transformation is given by $\sigma^{\mathrm{cross}}[s] = \sigma[s + 2iK']$. Since $\sigma[s + 8iK'] = \sigma[s]$, the branch points $u = \pm 2g \pm i/2$ may be of the fourth or the second order. We assume in the following that they are of square root type which is compatible with the analytical structure of the Bethe Ansatz equations [6].

The dressing factor $\sigma[u, v]$ can be represented in the following form [9, 24]:

$$\sigma[u, v] = e^{i\theta[u,v]}, \qquad \theta[u, v] = \chi[x^+, y^-] - \chi[x^-, y^-] - \chi[x^+, y^+] + \chi[x^-, y^+], \tag{3}$$

where $\chi[x, y]$ has the symmetry $\chi[x, y] = -\chi[y, x]$. To fix the solution of (2), we assume that χ is an analytic single-valued function for $|x| > 1$. Since the Jukowsky map (1) resolves the simple branch points of $\sigma[u, v]$ at $u = \pm 2g \pm i/2$, the function $e^{i\chi[x,y]}$ is meromorphic in the vicinity of the points $x = \pm 1$.

The crossing transformation acts on the functions $x^\pm$ by inverting them: $x^\pm \to 1/x^\pm$. Since $\chi[x, y]$ may not be single valued in the domain $|x| < 1$, the function σ^{cross} will be written in terms of the functions $\chi[1/x, y]$ defined on different Riemann sheets of the x-plane. To avoid this problem we analytically continue the crossing equation (2) along the path A^*B to the region with mirror kinematics. The crossing equation at the point B is written as

$$\sigma^{A^*B}[u, v]\sigma^{AB}[u, v] = \frac{1 - \frac{1}{x^+ y^+}}{1 - \frac{1}{x^- y^-}} \frac{1 - \frac{1}{x^- y^+}}{1 - \frac{1}{x^+ y^-}}, \tag{4}$$

where A^*B and AB denote the paths which were used for analytical continuation of $\sigma[u, v]$.

The point B is not necessarily at the same position as the points A and A^* in the u-plane. The position of the point B can be chosen in such a way that AB and A^*B correspond to the shifts of the s variable, $s \to s \pm iK'$. These shifts relate the theory with its mirror [25].

In the following, we will use the functions $\sigma_1[x, v]$ and $\sigma_2[x, y]$ given by

$$\sigma_1[x, v] = e^{i\chi[x,y^-] - i\chi[x,y^+]}, \qquad \sigma_2[x, y] = e^{i\chi[x,y]}. \tag{5}$$

Since the paths A^*B and AB cross only one simple branch cut, we are allowed to write[1]

$$\sigma^{A^*B} = \frac{\sigma_1[x^+, v]}{\sigma_1[1/x^-, v]}, \qquad \sigma^{AB} = \frac{\sigma_1[1/x^+, v]}{\sigma_1[x^-, v]}. \tag{6}$$

Now all four functions σ_1 which are used in (6) are on the same Riemann sheet of the x-plane.

For the further analysis, it will be convenient to write the crossing equation (4) in terms of the shift operator

$$D \equiv e^{\pm \frac{i}{2}\partial_u} : f[u] \mapsto f[u \pm i/2], \tag{7}$$

so that $x^+ = Dx$ and $x^- = D^{-1}x$. Namely, introducing the notation

$$f^{D^{\pm 1}} \equiv e^{D^{\pm 1}\log[f]}, \tag{8}$$

we can write (4) as

$$(\sigma_1[x, v]\sigma_1[1/x, v])^{D - D^{-1}} = \left(\frac{x - \frac{1}{y^+}}{x - \frac{1}{y^-}}\right)^{D + D^{-1}}. \tag{9}$$

The shift operator D is not well defined inside the strip $|\text{Re}[u]| \leqslant 2g$ since we can cross the cut of $x[u]$ and go to another sheet. To avoid this ambiguity we will consider the crossing equation (9) outside this strip, solve it, and then analytically continue the solution.

3. Solution

The function $\sigma_1[x, v]\sigma_1[1/x, v]$ as a function of u does not have a branch cut $[-2g, 2g]$. A solution of (9) with this property is given by

$$\sigma_1[x, v]\sigma_1[1/x, v] = \left(\frac{x - \frac{1}{y^+}}{x - \frac{1}{y^-}}\right)^{-\frac{D^2}{1-D^2} + \frac{D^{-2}}{1-D^{-2}}}, \tag{10}$$

$$\frac{D^{\pm 2}}{1 - D^{\pm 2}} = D^{\pm 2} + D^{\pm 4} + \cdots.$$

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[1] Strictly speaking, we also have to demand absence of the branch points between the cuts $|x^-| = 1$ and $|x^+| = 1$. This demand is analogous to the demand in relativistic field theories for the S-matrix to be meromorphic in terms of the rapidity variable.

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Strictly speaking, this expression should be regularized to have a precise meaning. However, the regulating terms will cancel for the complete dressing factor $\sigma[u, v]$.

Expression (10) can be further simplified using the fact that $\sigma_1[x, v] = \sigma_2[x, y^+]/\sigma_2[x, y^-]$:

$$\sigma_2[x, y]\sigma_2[1/x, y] = \left(\frac{x - \frac{1}{y}}{\sqrt{x}}\right)^{-\frac{D^2}{1-D^2} + \frac{D^{-2}}{1-D^{-2}}}. \tag{11}$$

The multiplier $1/\sqrt{x}$ does not contribute to (10). It is needed for the consistency with the antisymmetry of $\chi[x, y]$ with respect to the interchange $x \leftrightarrow y$. Indeed, a direct calculation shows that

$$\sigma_2[x, y]\sigma_2[1/x, y]\sigma_2[x, 1/y]\sigma_2[1/x, 1/y] = (u - v)^{-\frac{D^2}{1-D^2} + \frac{D^{-2}}{1-D^{-2}}} = \frac{\Gamma[1 - i(u - v)]}{\Gamma[1 + i(u - v)]}, \tag{12}$$

whose logarithm is antisymmetric with respect to $u \leftrightarrow v$ as it should be.

The ratio of gamma-functions resembles the solutions of the crossing equation in the relativistic theories. Note that the typical dressing factor in the relativistic theories is given by an expression of the type $\theta^{-\frac{D^2}{1+D^2} + \frac{D^{-2}}{1+D^{-2}}}$ [26], i.e. with the opposite sign in the denominator.

By taking the logarithm of (12), we get a simple Riemann–Hilbert problem which is solved by

$$\chi[x, y] = -i\tilde{K}_u \tilde{K}_v \log\left[\frac{\Gamma[1 - i(u - v)]}{\Gamma[1 + i(u - v)]}\right], \tag{13}$$

with the kernel $\tilde{K}$ defined by

$$(\tilde{K} \cdot F)[u] = \int_{-2g+i0}^{2g+i0} \frac{dw}{2\pi i} \frac{x - \frac{1}{x}}{z - \frac{1}{z}} \frac{1}{w - u} F[w]. \tag{14}$$

The kernel $\tilde{K}$ is constructed to satisfy the following equation[2]:

$$(\tilde{K} \cdot F)[u + i0] + (\tilde{K} \cdot F)[u - i0] = F[u], \qquad u^2 < 4g^2. \tag{15}$$

The subscripts u and v in (13) refer to action of $\tilde{K}$ on u and v variables respectively.

The solution (13) was chosen among the other possible solutions by the requirements that $\chi[x, y]$ should be analytic in the physical domain $|x| > 1$ and $\chi[x, y] \to$ const, $x \to \infty$.

Expression (13) can be rewritten in the form proposed by Dorey, Hofman and Maldacena [28] if we rewrite the action of the kernel $\tilde{K}$ as an integral in the Jukowsky plane:

$$(\tilde{K} \cdot F)[u] = \oint_{|z|=1} \frac{dz}{2\pi i} \frac{1}{x - z} F\left[g\left(z + \frac{1}{z}\right)\right] + \text{const} \tag{16}$$

and note that the constant term does not contribute to the dressing phase. This kind of transformation was discussed in [21].

Therefore, we see that the solution we obtained is nothing but the BES/BHL dressing phase.

Solution (13) of the crossing equation is not unique. It can be thought of as a minimal solution in the sense that we chose the solution with a minimal possible number of singularities. The form of the CDD factors can be strongly constrained assuming the decomposition (3), square root type of the branch points and the absence of branch points different from the ones

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[2] Interestingly, this equation appears for the resolvent of the Bethe roots condensate in the principal chiral field [27]. There its solution naturally leads to the AFS dressing phase.
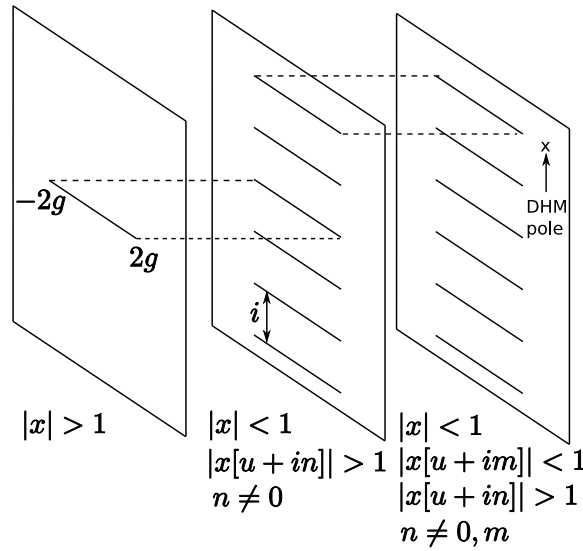
Figure 2. Analytical structure of $\sigma_2[x, y]$ as a function of u.

4. Analytical structure

that are present in the BES/BHL dressing phase. Then the CDD factor should satisfy the equation

$$(\sigma_{1,\text{CDD}}[x, v]\sigma_{1,\text{CDD}}[1/x, v])^{D-D^{-1}} = 1. \tag{17}$$

We see that the function $f_{\text{CDD}}[u] = \sigma_{1,\text{CDD}}[x, v]\sigma_{1,\text{CDD}}[1/x, v]$ should be periodic with the period i. Since by construction $f_{\text{CDD}}[u]$ does not have a branch cut $[-2g, 2g]$, due to periodicity it cannot have other branch cuts as well. Therefore f_{CDD} is a meromorphic function of u and $\sigma_{1,\text{CDD}}[x, v]$ is a meromorphic function of x. As a consequence, all the branch points of the dressing factor are resolved by introducing an elliptic parametrization. An arbitrary CDD factor is therefore a meromorphic function on the torus which satisfies the condition

$$\sigma_{\text{CDD}}[s]\sigma_{\text{CDD}}[s + 2iK'] = 1. \tag{18}$$

4. Analytical structure

The investigation of the analytical structure of the solution (13) is based on the property (15). It is instructive to write (13) as

$$-i\chi = \tilde{K}_u \left(\frac{D^2}{1 - D^2} - \frac{D^{-2}}{1 - D^{-2}} \right) \tilde{K}_v \log[u - v]. \tag{19}$$

Let us consider χ as a function of u. We will keep the notation χ to denote $\chi[x[u], y]$ in the domain $|x| > 1$. In this domain χ is analytic everywhere except on the Jukowsky cut $|x| = 1$. Crossing this cut will bring us to the other Riemann sheet. We will denote the function χ on this sheet by χ'. From (15) and (19) we deduce that

$$-i\chi' = i\chi + \sum_{n \neq 0} \text{sign}[n]\tilde{K}_v \log[u - v + in]. \tag{20}$$

We see that χ' has an infinite set of simple branch cuts given by the condition $|x[u + in]| = 1$. By construction χ' is defined in the domain $|x[u]| < 1$ and $|x[u + in]| > 1$ for $n \neq 0$.

Passing through one of the cuts $|x[u + \mathrm{i}m]| = 1$ with $m \neq 0$ brings us to yet a new Riemann sheet which is defined by $|x| < 1$ and $|x[u + \mathrm{i}m]| < 1$. The corresponding function χ_m'' is given by

$$-\mathrm{i}\chi_m'' = \mathrm{i}\chi' + \mathrm{sign}[m] \log \left[\frac{u - v + \mathrm{i}m}{-y} \right]. \tag{21}$$

The second term on the rhs of (21) leads us to the DHM poles described in [28]. These poles should be squared in the scattering matrix since the dressing factor contributes as σ^2.

The analytical properties of the dressing phase follow from the analytical properties of the function χ and the decomposition (3). Note however that from the crossing equation (2) it follows that on the Riemann sheet which contains the point A^* the dressing phase should have only two cuts $|x^\pm| = 1$. The cancelations of the other cuts[3] on this sheet between $\sigma_2[x^+, y]$ and $\sigma_2[x^-, y]$ can be easily seen from (20).

The representation (19) is closely related to the representation for the dressing kernel in [23, 29]. As is shown in [29], the dressing kernel can be nontrivially simplified at strong coupling. The integral representation of the dressing kernel can also be obtained by an inverse half-Fourier transform [21] of the dressing kernel of the BES equation [13].

Acknowledgments

Note added. After the work on this project was already finished, the paper [30] appeared. In [30] it was shown that the BES/BHL dressing phase satisfies the crossing relation (2). In contrast to [30], here we derive the dressing phase in a constructive way. Also in [30] the analytic structure of the dressing phase as a function on the torus is discussed. Here we present the analytic structure of the dressing phase as a function of u.

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[3] These cancelations were also independently observed by N Gromov and P Vieira.

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